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Self-Consistent Calculation of the ρ-Meson Regge Pole

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The left-hand discontinuities in the partial-wave amplitudes for π-π scattering are assumed to be dominated by the exchange of the ρ meson in a form suggested by the Regge representation for a resonance. This Regge behavior provides the necessary high-energy cutoff and allows the N/D equations to be solved. The partial-wave $I=1$ amplitudes are calculated for noninteger angular momenta $l<1$ as well as $l=1$. The trajectory $\alpha_\rho(s)$ as well as the residue $\beta_\rho(s)$ of the ρ-meson Regge pole are evaluated. An attempt is made to obtain a self-consistent solution for the relevant parameters, namely the position and width of the ρ resonance and $\alpha_\rho(0)$. The results of this calculation give $\alpha_\rho(0)\gtrsim 0.9$. The $I=0$ vacuum trajectory is also discussed.

I. INTRODUCTION

THERE have been a number of papers written on the problem of determining the position and width of the ρ meson self-consistently.[1,2] In essence, these bootstrap calculations of the ρ used the exchange of this $I=1$, $l=1$ resonance in the crossed channels to provide the force necessary to produce the ρ meson in the direct channel. The $l=1$ part of the interaction is projected out and the partial-wave dispersion relations are solved by the N/D method. The hope is that the solution yields a resonance having the same position and width as that of the exchanged one.

A major difficulty is due to the divergence arising from the exchange of a massive vector particle, with sufficiently large coupling, which necessitates the use of a cutoff. Instead of considering the ρ to be a vector particle even when the energy of the exchanged ρ is not close to the resonant energy, Wong[2] employed a form suggested by the Regge representation for a resonance. This then provides a cutoff at high energy, the relevant parameter being the angular momentum of the ρ trajectory at zero energy, $\alpha_\rho^{\mathrm{In}}(0)$.

The purpose of this article is to carry Wong's ρ (bootstrap) calculation with a "Regge cutoff" a step further. For $l=1$ we carry out a calculation similar to his but then continue the N/D equations for noninteger angular momenta and calculate $\alpha_\rho(s)$, comparing $\alpha_\rho(0)$ with the input parameter $\alpha_\rho^{\mathrm{In}}(0)$. In other words, this is an attempt to bootstrap not only the position and width of the ρ resonance, but the slope of its Regge trajectory. The residue function $\beta_\rho(s)$ is also determined. The sensitivity of our results to some of the approximations made is examined. For example, the above calculation is compared to a similar one in which we take the exchanged ρ to have constant angular momentum and employ a straight cutoff. The $I=0$ vacuum trajectory is also calculated.

Section II is devoted to a presentation of the relevant formalism. The results of the numerical calculations are given and discussed in Sec. III.

The results may be summarized as follows: In the same sense that the usual bootstrap calculations of the ρ are not self-consistent, i.e., the output width of the ρ (for reasonable values of the position of the ρ) is larger than the input width of the exchanged ρ,[1,2] so the calculated $\alpha_\rho(0)$ is larger than the input parameter $\alpha_\rho^{\mathrm{In}}(0)$. For all cases, both $\alpha_\rho(0)$ are $\gtrsim 0.9$, in agreement with results of Foley *et al.*[3] and the calculation of Chang and Sharp[4]; however, in disagreement with other determinations of $\alpha_\rho(0)\sim 0.5$.[5] The residue of the ρ Regge trajectory, after removal of a threshold factor, turns out to be nearly constant in the scattering region ($s<0$) and very close to the input β. The calculations of the $I=0$ vacuum pole trajectory give a small slope: $\alpha_P'(0)\lesssim 1/500$.[6]

II. FORMULATION OF THE INTEGRAL EQUATIONS

We shall obtain amplitudes for pion-pion scattering by the familiar N/D solution[7] of the partial-wave dispersion relations. The usual expressions for the scalar variables s, t, and u in terms of the momentum k and scattering angle θ in the center-of-mass system of the direct or s channel are[6] $s=4(k^2+1)$, $t=-2k^2(1-\cos\theta)$, and $u=4-s-t$. The invariant partial-wave amplitude A_l is defined in terms of the S matrix by

$$A_l(s)\equiv(1/2i\rho)(S_l-1)\equiv B_l(s)+{}^RA_l(s)\,, \qquad (1)$$

where

$$\rho=((s-4)/s)^{1/2}\,, \qquad (2)$$

and B_l is regular for $s>0$ and ${}^RA_l(s)$ has only a right-

* Supported by the U. S. Atomic Energy Commission.
† Supported in part by the U. S. Air Force through Air Force Office of Scientific Research Grant AF-AFOSR-62-452.
[1] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[2] D. Wong, Phys. Rev. **126**, 1220 (1962).
[3] K. Foley, S. Lindenbaum, W. Love, S. Ozaki, J. Russell, and L. Yuan, Phys. Rev. Letters **10**, 376 (1963).
[4] H. Cheng and D. Sharp, Phys. Rev. **132**, 1854 (1963).
[5] I. Muzinich, Phys. Rev. Letters **11**, 88 (1963).
[6] We use units $\hbar=c=m_\pi=1$.
[7] G. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

hand cut. The right-hand discontinuity in $A_l(s)$ is given by unitarity: We make the approximation that elastic unitarity holds for all physical k^2:

$$A_l(s)=B_l(s)+\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}|A_l(s')|^2\left(\frac{s'-4}{s'}\right)^{1/2}. \quad (3)$$

The left-hand discontinuity or generalized potential[8] is derived from application of an approximate form of crossing symmetry. We will first determine $B_l(s)$ and then discuss the N/D equations and their solution.

Using crossing symmetry, $B_l(s)$ is calculated from the scattering amplitude in the crossed t and u channels. We will consider *only* the exchange of the $I=1$ ρ resonance in the t and u channels. Then in the s channel for $I=1$ and l equal to an *integer* we obtain

$$B_l^{I=1}(s)=\frac{1}{2}\int_{-1}^{1} P_l(\cos\theta)d\cos\theta \times[\tfrac{1}{2}A_R^{I=1}(t,s)-\tfrac{1}{2}A_R^{I=1}(u,s)], \quad (4)$$

which for l odd becomes

$$B_l^1(s)=\frac{1}{(s-4)}\int_{-(s-4)}^{0} P_l\left(1+\frac{2t}{s-4}\right)dtA_R^1(t,s), \quad (5)$$

where $A_R^1(t,s)$ is the part of the scattering amplitude in the t channel, $A^1(t,s)$, which has no singularities for $s>0$, i.e., $t<4$.

Taking a Breit-Wigner form for the ρ resonance, we have

$$A^1(t,s)\approx\frac{3\Gamma(t-4)}{m_\rho^2-t-i\Gamma(t-4)^{3/2}/t^{1/2}}P_1\left(1+\frac{2s}{t-4}\right). \quad (6)$$

Further making the narrow width approximation, so that $A_R^1(t,s)=A^1(t,s)$, we have the simple form for l equal to an odd integer[9]:

$$B_l^1(s)=\frac{6\Gamma}{s-4}(m_\rho^2-4+2s)Q_l\left(1+\frac{2m_\rho^2}{s-4}\right). \quad (7)$$

Equation (7) has an acceptable behavior in the l plane as $|l|\to\infty$ and thus can be continued for noninteger l even though both (4) and (5) cannot.[10] However, $B_l(s)$ as given by (7) diverges like $\log(s)$ as $s\to\infty$ and the resulting N/D equations do not have a unique solution.

A mechanism that damps this singular high-energy behavior is provided by the Regge motion of resonance poles. In the Regge description for the ρ resonance we take

$$A^1(t,s)=\frac{b_\rho(t)}{\sin\pi\alpha_\rho(t)}\frac{1}{2}\left[P_{\alpha_\rho(t)}\left(-1-\frac{2s}{t-4}\right)-P_{\alpha_\rho(t)}\left(1+\frac{2s}{t-4}\right)\right]. \quad (8)$$

We are interested in B_l for $s\geq 4$ and hence in the region $t\leq 0$ where $\alpha_\rho(t)$ is real and <1. For large s, (8) is or order $s^{\alpha_\rho(t)}$ and hence an acceptable input to the N/D equations.

Since we do not know the behavior of $b_\rho(t)$ or $\alpha_\rho(t)$ except in the immediate vicinity of the ρ resonance, we will take a very simple form for (8) which reduces to the correct Breit-Wigner form (6) near $t=m_\rho^2$, yields the same $B_{l=1}^1(s=4)$ as Eq. (7), and gives the same high-energy behavior in s (for small t) as the Regge pole:

$$A_R^1(t,s)\approx\frac{3\Gamma(t-4)}{(m_\rho^2-t)}\left(1+\frac{2s}{t-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho'(0)(t-m_\rho^2)}. \quad (9)$$

With this approximation, $A_{l=1}^1(s)$ is readily calculated numerically.[11] However we are interested in continuing the partial-wave amplitude for noninteger l. Eq. (5) *cannot* be continued; there are alternate formulations for $B_l(s)$ which can be continued.[10] From the point of making the computations manageable, we again note that *expression (7) can be continued in the l plane.* Thus we are led to make the further approximation that using (5) in making the partial-wave projection $B_l^1(s)$ of (9) we evaluate the last factor $(s/4)^{\alpha_\rho'(0)(t-m_\rho^2)}$ at $t=0$ (where it gives the maximum contribution). Hence our "Reggeized" $B_l^1(s)$ becomes[12]

$$B_l^1(s)=\frac{6\Gamma}{(s-4)}(m_\rho^2-4+2s)Q_l\left(1+\frac{2m_\rho^2}{s-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho(0)-1}. \quad (10)$$

This expression which is our approximate form for the left-hand cut for the partial wave $\pi-\pi$ amplitude in the $I=1$ state and odd integer l has acceptable behavior for large l and *can* be continued in the l plane.

Now in order to insure that $A_l^1(s)$ has the proper threshold behavior, i.e., $(s-4)^l$ and also remove this additional cut from $B_l^1(s)$ for noninteger l, we define new amplitudes

$$\mathbf{A}_l^1(s)\equiv 1/(s-4)^l A_l^1(s)\equiv 1/2i\rho_l(S_l-1) \equiv \mathbf{B}_l^1(s)+{}^R\mathbf{A}_l^1(s), \quad (11)$$

where

$$\rho_l=((s-4)/s)^{1/2}(s-4)^l, \quad (12)$$

[8] G. Chew and S. Frautschi, Phys. Rev. **124**, 264 (1961).

[9] If we look at $I=0$ and even angular momenta, the relevant Born term is of the same form as (7) with Γ replaced by 2Γ.

[10] E. Squires, Nuovo Cimento **25**, 242 (1962). A continuation of Eq. (5) based on the lines discussed in this reference will yield the same result.

[11] Equation (9) and other more complicated approximations to (8) were considered and used to calculate $A_{l=1}^1(s)$ even though these could not be continued to noninteger l simply.

[12] Thus the input cutoff parameter $\alpha_\rho^{\mathrm{In}}(0)$ should be considered as some average value. Using form given by (8) would have necessitated a somewhat smaller $\alpha_\rho^{\mathrm{In}}(0)$.

and

$$\mathbf{B}_l{}^1(s)=\frac{6\Gamma}{(s-4)^{l+1}}(m_\rho{}^2-4+2s)\times Q_l\left(1+\frac{2m_\rho{}^2}{s-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho(0)-1}. \quad (13)$$

Now define

$$\mathbf{A}_l{}^1(s)=N_l(s)/D_l(s)\,, \quad (14)$$

where N has only a left-hand cut and D has only a right-hand cut. Then in terms of the generalized potential $\mathbf{B}_l{}^1(s)$ which is regular in the physical region, the N and D equations are [2,13]

$$D_l(s)=1-(s-s_0)\frac{P}{\pi}\int_4^\infty \rho_l(s')N_l(s')\frac{ds'}{(s'-s)(s'-s_0)} -i\rho_l(s)N_l(s)\Theta(s-4)\,, \quad (15)$$

$$N_l(s)=\mathbf{B}_l{}^1(s)+\frac{1}{\pi}\int_4^\infty\left(\mathbf{B}_l{}^1(s')-\frac{(s-s_0)}{(s'-s_0)}\mathbf{B}_l{}^1(s)\right)\times\rho_l(s')N_l(s')\frac{ds'}{s'-s}. \quad (16)$$

Note that the solutions $\mathbf{A}_l{}^1(s)$ are independent of the subtraction point s_0. As long as $0<l<2-\alpha_\rho(0)<2$, these equations have unique solutions. The Fredholm integral Eq. (16) for $N_l(s)$ was solved by matrix inversion on the Stanford 7090 computer.

For given input parameters $m_\rho{}^{\text{In}}$, Γ^{In} and $\alpha_\rho{}^{\text{In}}(0)$, which determine $B_l{}^1(s)$ [$\alpha_\rho{}^{\text{In}}(0)$ being fixed by the requirement that we get an $l=1$ resonance at $m_\rho{}^{\text{In}}$, i.e., $\text{Re}D_{l=1}(s=(m_\rho{}^{\text{In}})^2)=0$], we calculate the width of the $l=1$ resonance. Then we solve (15) and (16) for non-integer $l<1$ in order to determine the properties of the ρ trajectory. For a given l, we look for the value of $s(\equiv s_l)$ for which $\text{Re}D_l(s)=0$:

$$\text{Re}D_l(s_l)=0. \quad (17)$$

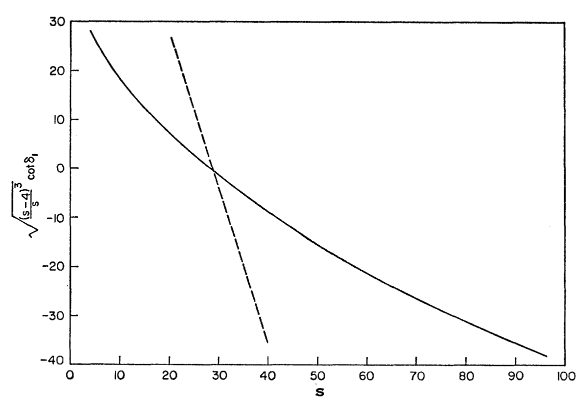

FIG. 1. Phase shift for $I=1$, $l=1$ amplitude versus s. The solid curve corresponds to the output, whereas the dashed curve comes from our input Breit-Wigner form. $\Gamma^{\text{In}}=0.145$ and $\alpha_\rho{}^{\text{In}}(0)=0.949$. For Figs. 1–4, $(m_\rho{}^{\text{In}})^2=29$ and the "cutoff parameter," i.e., $\alpha_\rho{}^{\text{In}}(0)$ is adjusted to force an $l=1$ resonance at $m_\rho{}^{\text{In}}$.

[13] J. Uretsky, Phys. Rev. **123**, 1459 (1961).

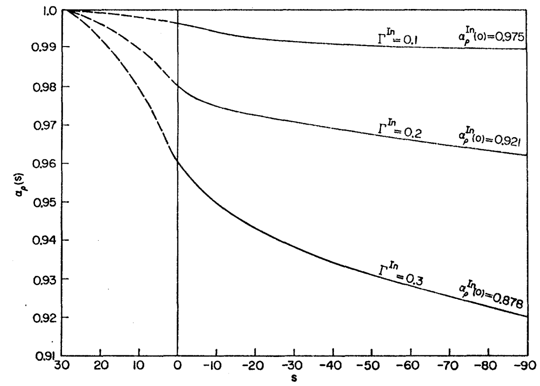

FIG. 2. $\alpha_\rho(s)$ for various input parameters. The dashed lines for $s>4$ in Figs. 2–4 emphasize that we only investigated the vanishing of the real part of $D_l(s)$.

For $s_l<4$ this gives directly the Regge trajectory $\alpha_\rho(s)$, whereas for $s_l>4$, in the limit of a narrow resonance, it gives approximately $\text{Re}\alpha_\rho(s)$. The residue $b_\rho(s)$ is determined as follows: Since $\text{Re}D_l(s_l)=0$, in the vicinity of s_l we have (for $s_l<4$)

$$\mathbf{A}_l{}^1(s)=\left(N_l(s)\Big/\frac{\partial\,\text{Re}D_l(s)}{\partial s}\right)_{s_l}\Big/(s-s_l). \quad (18)$$

This residue is real since $N_l(s_l)$ is simply given by

$$N_l(s_l)=\frac{P}{\pi}\int_4^\infty \mathbf{B}_l{}^1(s')\rho(s')N_l(s')\frac{ds'}{s'-s_l}. \quad (19)$$

The partial-wave projection of the ρ Regge pole of "odd j parity"[10] divided by the threshod factor $(s-4)^l$,

$$\frac{b_\rho(s)}{\sin\pi\alpha_\rho(s)(s-4)^l}P_{\alpha_\rho(s)}\left(-1-\frac{2t}{s-4}\right) \equiv\frac{\beta_\rho(s)\pi(2\alpha_\rho(s)+1)}{\sin\pi\alpha_\rho(s)}P_{\alpha_\rho(s)}\left(-1-\frac{2t}{s-4}\right), \quad (20)$$

then must be compared with (18).[14] Now

$$\frac{1}{2}\int_{-1}^1 P_l(\cos\theta)P_{\alpha_\rho(s)}(-\cos\theta)d\cos\theta\beta_\rho(s)\frac{\pi(2\alpha_\rho(s)+1)}{\sin\pi\alpha_\rho(s)} =\frac{\beta_\rho(s)(2\alpha_\rho(s)+1)}{(\alpha_\rho(s)-l)(\alpha_\rho(s)+l+1)}\underset{s\approx s_l}{\approx}\frac{\beta_\rho(s_l)}{\alpha_\rho'(s_l)(s-s_l)}. \quad (21)$$

Thus for a given l, we find $\alpha'(s_l)$ from $\alpha(s)$ [as found from (17)] and hence the residue is given by

$$\beta_\rho(s_l)=\left(N_l(s)\Big/\frac{\partial\,\text{Re}D_l(s)}{\partial s}\right)_{s_l}\alpha_\rho'(s_l). \quad (22)$$

[14] For the evaluation of the residue function $b_\rho(s)$ we have factored out the threshold factor $(s-4)^l$. The partial-wave projection of a single Regge pole does not have this (correct) threshold behavior but goes as $(s-4)^{\alpha_\rho(s)}$. As $\alpha_\rho(s)$ varies little over a large range of s, the above definition of β is adequate. A representation in which each pole has the correct partial-wave threshold behavior has been given by N. Khuri, Phys. Rev. **130**, 429 (1963).

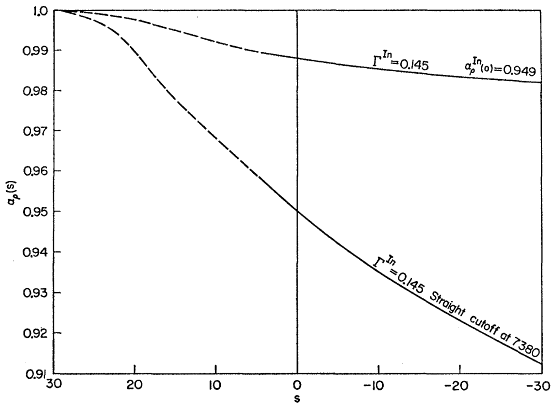

FIG. 3. Comparison of $\alpha_\rho(s)$ for a "straight cutoff" and a "Regge cutoff."

III. RESULTS AND CONCLUSIONS

As discussed earlier, in addition to evaluating the $I=1$, $l=1$ $\pi-\pi$ scattering amplitude in an attempt to "bootstrap" the ρ meson, we calculate the ρ's Regge pole parameters for noninteger $l<1$. We computed both the position α_ρ and residue β_ρ of the pole as functions of s.

We investigated the problem for several values of the input coupling constant Γ^{In} (or input width of the ρ) and for several input masses $(m_\rho^{\text{In}})^2$ ranging from 10 to the experimental value of 29. No self-consistent solution was obtained. The procedure was to evaluate the $I=1$, $l=1$ amplitude for many values of $\alpha_\rho^{\text{In}}(0)$ until the mass of the input ρ was reproduced by a zero of $\text{Re}D_{l=1}(s)$ at $s=(m_\rho^{\text{In}})^2$, i.e., we always forced the mass of the produced ρ to be the same as that of the exchanged ρ. The output width could be determined either by evaluating the quantity $[N_{l=1}(s)/\partial D_{l=1}(s)/\partial s]$ at the position of the resonance (which is a correct procedure for a narrow resonance), or by actually looking at the $l=1$ phase shift as a function of s. In either the former case or the latter looking below the resonant energy the output width was larger than the input one by a factor of 3–6. Looking at the phase shift itself on the high-energy side of the resonance situation is even worse. The function $((s-4)^3/s)^{1/2}\cot\delta_1(s)$ is plotted in Fig. 1 together with the input value for this function. For energies larger than the position of the ρ resonance the function decreases too slowly for a resonant behavior. The input values for the exchanged ρ were $(m_\rho^{\text{In}})^2=29$ and $\Gamma^{\text{In}}=0.145$ (which corresponds to a full width at half-maximum of 110 MeV).

Hence for given m_ρ^{In} and Γ^{In}, $\alpha_\rho^{\text{In}}(0)$ is determined from the self-consistency requirement on m_ρ in the $l=1$ calculation. Thus the generalized potential $\mathbf{B}_l^1(s)$ is determined and we solve the full N_l/D_l Eqs. (15) and (16) to determine the Regge trajectory and residue for the ρ. In Figs. 2 to 4 we present some of the results for $(m_\rho^{\text{In}})^2=29$. As the width of the produced ρ meson is rather large, the imaginary parts of the ρ trajectory will be large above $s=4$. Since we have only looked for the zero of the real part of D_l, we have obtained the actual trajectory only for $s<4$. We emphasize this by plotting dashed curves for $s>4$, e.g., the dashed $\alpha_\rho(s)$ curves correspond to an approximation to the real part of $\alpha_\rho(s>4)$.

For $\Gamma^{\text{In}}=0.145$ we show in Fig. 3 a comparison of α_ρ for a calculation as mentioned above to one in which a pure $l=1$ ρ exchange [as given by Eq. (7)] was considered as a straight cutoff used in solving Eqs. (15) and (16) (again the self-consistency requirement of the output ρ position equaling m_ρ^{In} determined the value of the cutoff). We see that although there is some quantitative difference, both trajectories have $\alpha_\rho(0)$ larger than 0.9. These calculations with the straight cutoff and other calculations specifically for $A_{l=1}{}^1(s)$, e.g., using (9) to calculate $\mathbf{B}_{l=1}{}^1(s)$,[11] all gave very similar results for the $l=1$ partial wave. We felt this was a fairly good test for a number of the approximations made in obtaining Eq. (10).

In addition to obtaining the output width larger than the input one, the output $\alpha_\rho(0)$ was larger than $\alpha_\rho^{\text{In}}(0)$.[12] The two discrepancies are correlated. Near the resonance, we have from (21), $(d\alpha_\rho/ds)=(\beta_\rho/\Gamma)$ so that a large Γ corresponds to a small slope for α and thus $\alpha_\rho(0)$ is larger at $s=0$ than $\alpha_\rho^{\text{In}}(0)$. It is interesting to note that the output β_ρ, as shown in Fig. 4, is almost constant in the relevant scattering region ($s<0$) and is very close in magnitude to $\beta_\rho^{\text{In}}=(d\alpha_\rho^{\text{In}}/ds)\Gamma^{\text{In}}$.

We have also calculated the scattering amplitude in $I=0$ channel again using only ρ exchange in the crossed channels. If we use the same parameters as for the $I=1$ calculation[9] we find that there is a vacuum trajectory but that for $s=0$ it has an $l>1$; specifically for $l=1$ the pole occurs for a very large negative s. Therefore we adjusted the cutoff parameters to force the $I=0$ trajectory to cross[15] 1 at $s=0$ and calculated the vacuum trajectory $\alpha_P(s)$. A typical curve is shown in Fig. 5. Note that the slope is quite small; $(d\alpha_P(s)/ds)_{s=0}\approx10^{-3}$

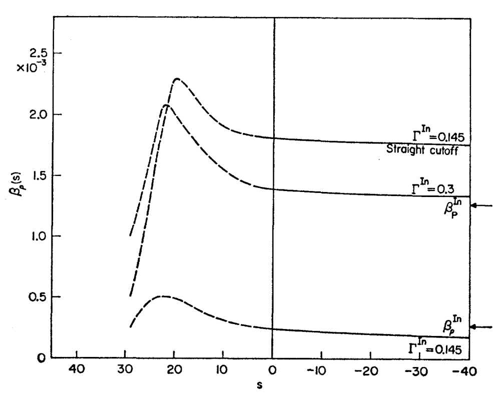

FIG. 4. The residue $\beta_\rho(s)$ for various input parameters. The arrows indicate the input $\beta_\rho^{\text{In}}=(d\alpha_\rho^{\text{In}}/ds)\Gamma^{\text{In}}$.

[15] If we then recalculate the $I=1$, $l=1$ amplitude, no ρ resonance occurs.

and hence our results would not be consistent with the f^0 [16] being on the vacuum trajectory. We also calculated the residue of the vacuum pole at $s=0$. The residue corresponding to the trajectory shown in Fig. 5 gave an asymptotic total $\pi-\pi$ cross section of 3 mb as compared to a value of the 15 mb obtained using the factorization theorem[17] and the asymptotic πN and NN cross sections.

We feel that both the problem (a) that the output ρ width is larger than the input ρ width and the problem (b) that using the input ρ parameters which yield a ρ resonance to calculate the $(I=0)$ vacuum trajectory give $\alpha_P(0)>1$ are largely due to the one channel approximation. The effect of an inelastic channel below its threshold is to, (i) always act as an attraction, and (ii) tend to narrow a resonance. Hence if we include the inelastic effects in the $I=1$ channel, which we expect to be due largely to the $\pi\omega$ channel,[1] this would narrow the output ρ width, and increase the attraction so that a somewhat smaller $\alpha_\rho^{\mathrm{In}}(0)$ would be required.[18] On the other hand, the $\pi\omega$ channel does not couple to the $I=0$ channel so that this additional attraction would not be present and hence we would have a smaller $\alpha_P(0)$.

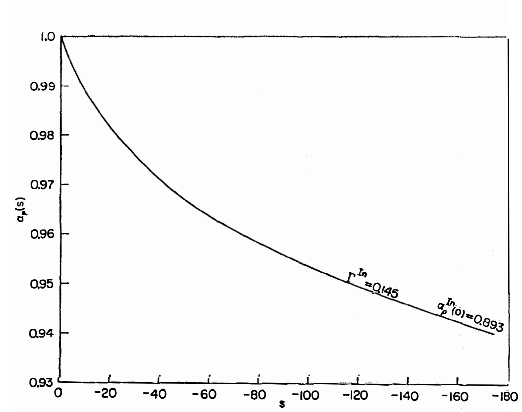

FIG. 5. The $I=0$ vacuum trajectory $\alpha_P(s)$ which has been adjusted to cross $s=0$ at $l=1$.

[16] W. Selove, V. Hagopian, H. Broad, A. Baker, and E. Leboy, Phys. Rev. Letters 9, 277 (1962).

[17] M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962); V. Gribov and I. Pomeranchuk, *ibid.* 8, 343 (1962).

[18] A relatively small change in $\alpha_\rho^{\mathrm{In}}(0)$ produces a large shift in the output resonance position.